II. Ray optics

Postulates

- Light travels in the form of rays. The rays are emitted by light sources and can be observed when they reach an optical detector.

- An optical medium is characterized by a quantity *n* called the *refractive index.* The refractive index is the ratio of the speed of light in free space *c* to that in the medium *c*/*n*. Therefore, the time taken by light to travel distance *d* equals *nd*/*c*. It is thus proportional to the product of *nd*, known as the *optical path length.*

- In an inhomogeneous medium the refractive index *n*(r) is a function of the position. The optical path length along a given path between two points *A* and *B* is therefore

$$\text{Optical path length} = \int_A^B n(\mathrm{r})\, ds \qquad \text{(II-1)}$$

  where *ds* is the differential element of the length along the path.

- **Fermat's principle.** Optical rays travelling between *A* and *B* follow a path such that the optical path length between the two points is an extremum, relative to neighbouring paths. The extremum may be a maximum, a minimum or a point of inflection. It is, however, usually a minimum, in which case *light rays travel along a path of least time.*

Fermat's principle dictates that in a homogeneous medium, light rays travel in straight lines, hence shadows are perfect projections of stops (Fig. II-1).

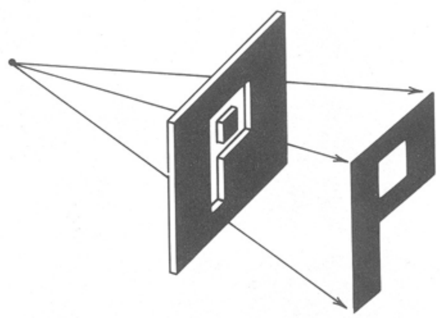


Fig. II-1

Reflection and refraction at plane surfaces

Reflection from a mirror or at the boundary between two media of different refractive index: the reflected ray lies in the plane of incidence, the angle of reflection equals the angle of incidence (Fig. II-2). As a consequence, *planar mirrors* reflect the rays originating from a point P_1 such that the reflected rays appear to originate from a point P_2 behind the mirror, called the image (Fig. II-3).

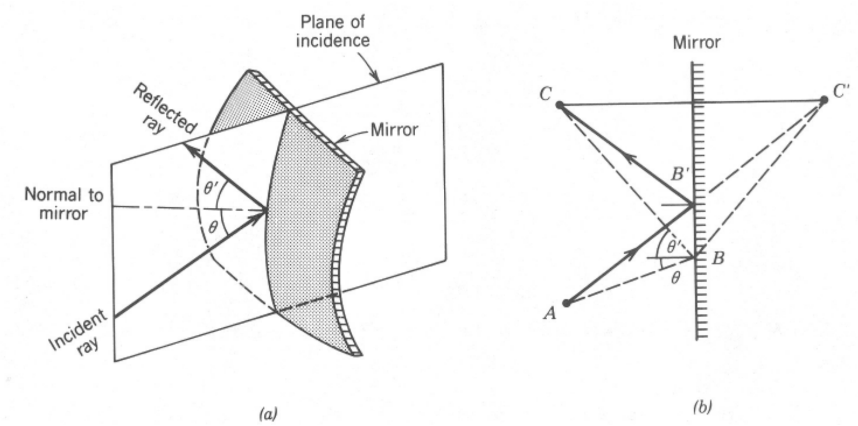


Fig. II-2

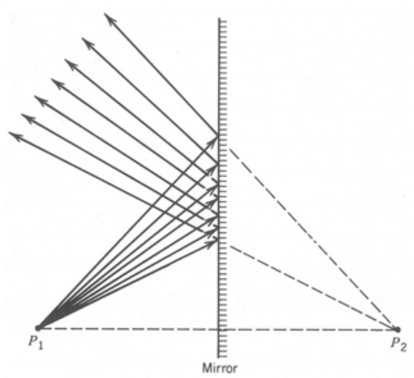


Fig. II-3

Refraction at the boundary between two media of different n: the refracted ray lies in the plane of incidence (Fig. II-4); the angle of refraction θ_2 is related to the angle of incidence θ_1 by Snell's law

$$n_1 \sin\theta_1 = n_2 \sin\theta_2 \qquad \text{(II-2)}$$

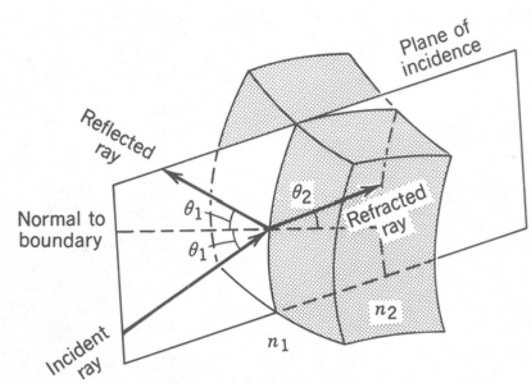


Fig. II-4

Exercise: Snell's law follows from Fermat's principle, as you can show by seeking to minimize the optical path length $n_1 d_{AB} + n_2\, d_{BC}$ between points *A* and *C* in Fig. II-5.

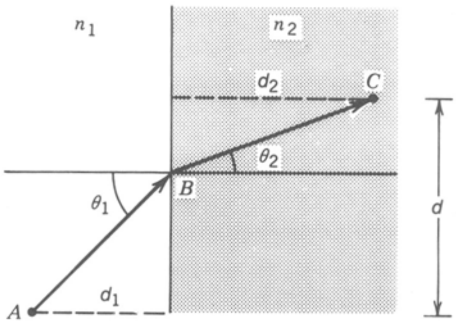


Fig. II-5

If the incident ray is in a medium of higher refractive index, $\theta_2 > \theta_1$, i.e. the refracted ray bends toward the boundary. For an incident angle

$$\theta_1 > \theta_c = \sin^{-1}\frac{n_2}{n_1} \qquad \text{(II-3)}$$

where θ_c is referred to as the critical angle, Snell's law (Eq. II-2) can not be satisfied and refraction does not occur. The incident ray is totally reflected (Fig. II-6) as if the surface were a perfect mirror (*total internal refraction*). The functioning of *reflecting prisms* and *optical fibres* are based on this effect (Fig. II-7).

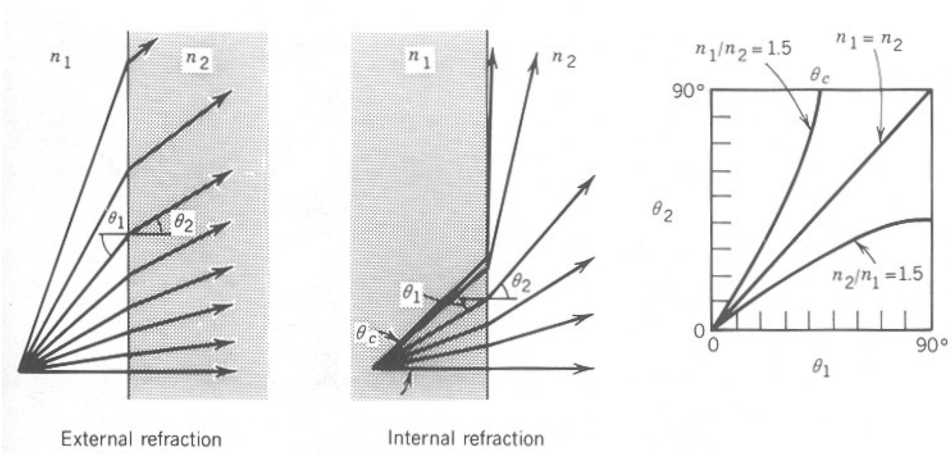


Fig. II-6

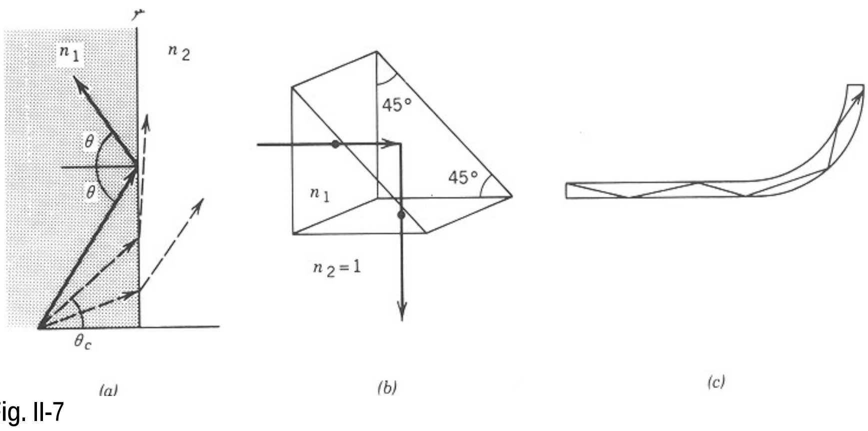


Fig. II-7

Focusing and imaging optics

A *paraboloidal mirror* has a surface formed by a paraboloid of revolution (Fig. II-8). It focuses all incident rays parallel to its axis to a single point called the *focus.* The distance d_{PF} is called the focal length. An *elliptical mirror* reflects all the rays emitted from one of its two foci, e.g. P_1, and collect them at the other focus, P_2 (Fig. II-9).

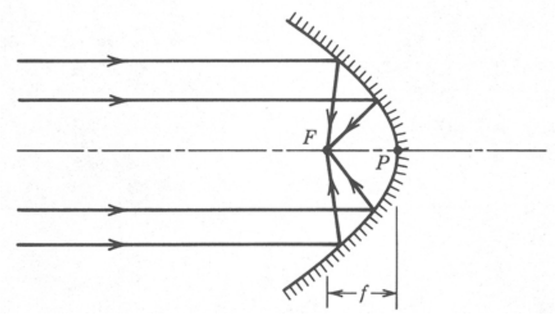


Fig. II-8

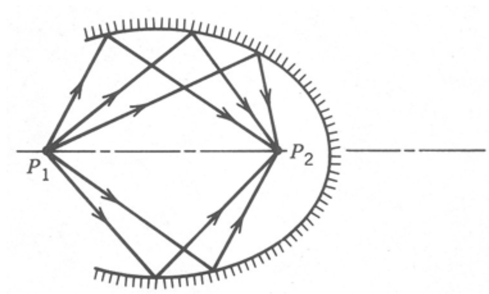


Fig. II-9

Spherical mirrors and lenses are easier to fabricate and align than a paraboloidal or elliptical mirror but their focusing and imaging capabilities are compromised, as it is apparent from the fact that incident rays parallel to the optics axis cross the axis at different points (Fig. II-10,11).

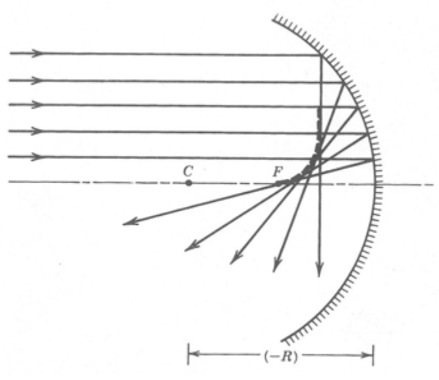


Fig. II-10

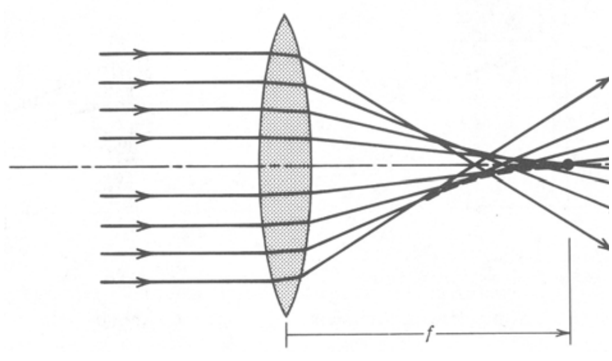


Fig. II-11

Rays that make small angles with the axis of the mirror or of the lens (such that $\sin\theta \approx \theta$) are called *paraxial rays.* In the *paraxial approximation* where only paraxial rays are considered, a spherical mirror has a focusing property like that of the paraboloidal mirror and an imaging property like that of the elliptical mirror. In the paraxial limit, for incident rays parallel to the mirror's axis, a spherical mirror of radius R acts like a paraboloidal mirror of focal length $f = -R/2$ by focusing them onto a single point F at a distance ($-R/2$) from the mirror centre C (Exercise). By convention, R is negative for a concave mirror and positive for convex mirrors, whereas rays originating from any point on the axis, e.g. P_1, are collected in a single corresponding point P_2 on the axis either by a mirror (Fig. II-12) or by a lens (Fig. II-13). The distances of the source and the image away from the mirror (or lens), z_1 and z_2, respectively, obey the imaging equation in both cases (Exercise):

$$\frac{1}{z_1}+\frac{1}{z_2}=\frac{1}{f} \qquad \text{(II-4)}$$

where for a lens f is determined by the radii of curvature of the surfaces and the refractive index of the prism material.

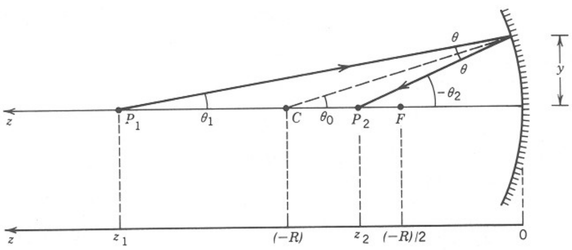


Fig. II-12

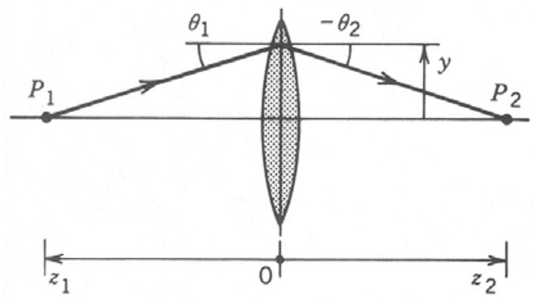


Fig. II-13

Rays originating from an off-axis point $P_1=(y_1,z_1)$ are reflected to a point $P_2=(y_2,z_2)$, see Fig. II-14, satisfying (Exercise)

$$y_2=-y_1\frac{z_2}{z_1} \qquad \text{(II-5)}$$

This means that rays from each point in the plane $z = z_1$ meet at a single corresponding point in the plane $z = z_2$, consequently the mirror (Fig. II-14) or the lens (Fig. II-15) *forms an image* of any object in the plane $z = z_1$ *with a magnification of* $-z_2/z_1$ in plane $z = z_2$.

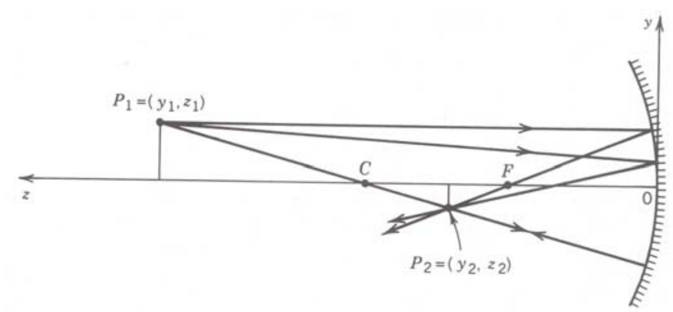


Fig. II-14

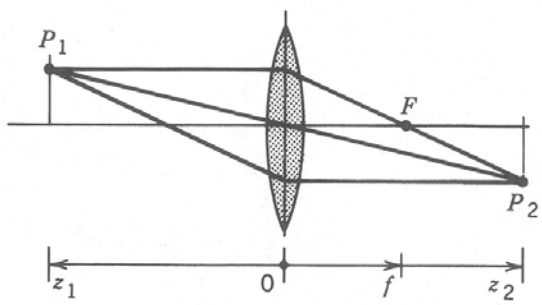


Fig. II-15

Light guides

Light can be guided from one location to another, in principle, by use of a set of lenses or mirrors (Fig. II-16). In practice, guiding light based on total internal reflection in optical fibres, allowed by $n_2 < n_1$ (Fig. II-17), turned out to be far superior to any other alternatives, thanks to the extremely low attenuation of fused silica fibres.

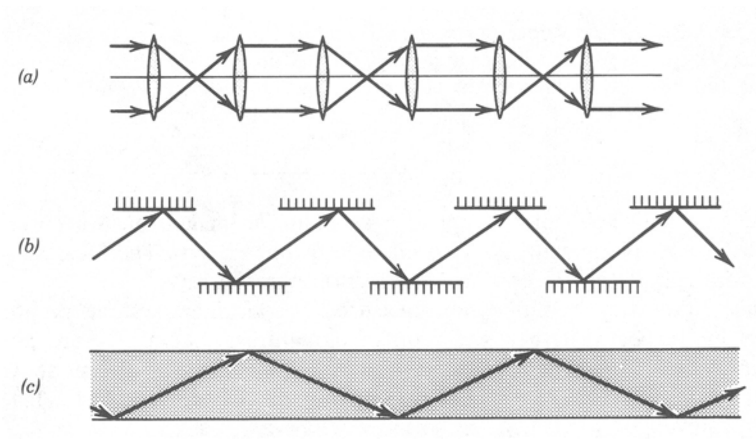


Fig. II-16

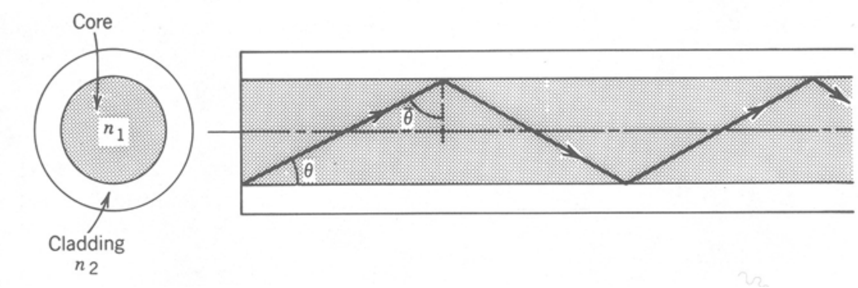


Fig. II-17

Whilst description of the distribution of light in and propagation of light through optical waveguides requires a refined model: the electromagnetic theory of light, ray optics can predict some properties important for practical applications such as the cone of rays an optical waveguide is able to accept and capture for guiding. The *numerical aperture* of the fibre is defined as the sine of the maximum angle (Fig. II-18) at which rays get captured and are subsequently guided in the core is given by

$$NA = \sin\theta_a = \sqrt{n_1^2 - n_2^2} \qquad \text{(II-6)}$$

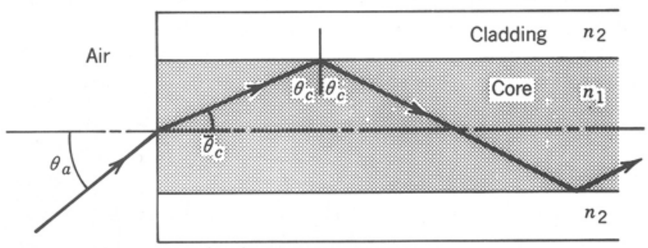


Fig. II-18

Matrix optics

Matrix optics is a technique for tracing paraxial rays. The rays are assumed to travel only within a single plane so that the formalism is applicable to systems with planar geometry and to meridional rays in circularly symmetric systems.

A ray is described by its position y and angle θ with respect to the optical axis (Fig. II-19). These variables are altered as the ray travels through the optical system (Fig. II-20). In the paraxial approximation, the position and angle at the input and output planes of an optical system are related by the *linear* algebraic equations

$$y_2 = A\,y_1 + B\theta_1 \qquad \text{(II-7a)}$$

$$\theta_2 = C\,y_1 + D\,\theta_1 \qquad \text{(II-7b)}$$

Consequently, the effect of the optical system on the incident rays is described by a 2x2 matrix called the *ray-transfer matrix*

$$\begin{bmatrix} y_2 \\ \theta_2 \end{bmatrix} = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{bmatrix} y_1 \\ \theta_1 \end{bmatrix} \qquad \text{(II-8)}$$

The ray-transfer matrix M, whose elements are *A, B, C, D,* characterizes the optical system completely (within the limits of ray optics and the paraxial approximation) because it permits (y_2,θ_2) to be determined for any (y_1,θ_1).

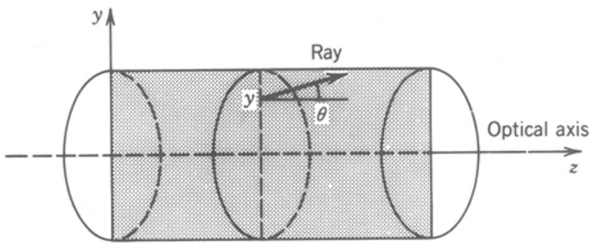


Fig. II-19

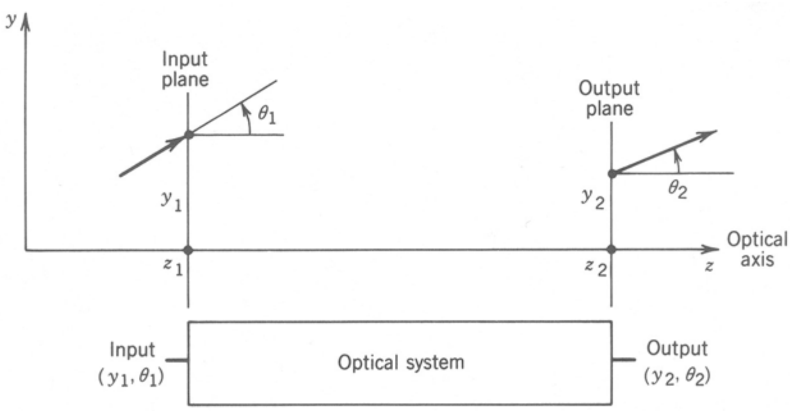
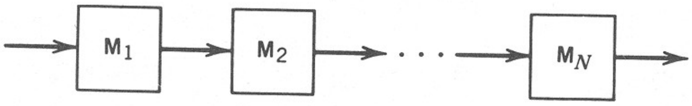


Fig. II-20

The convenience of using this matrix method lies in the fact that the ray-transfer matrix of a cascade of optical components is a product of the ray-transfer matrices of the individual components (Fig. II-21). Matrix optics therefore provides a powerful means of describing how complex optical systems transfer paraxial rays.

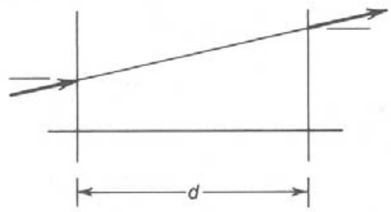


$$\mathbf{M} = \mathbf{M}_N \cdots \mathbf{M}_2\mathbf{M}_1.$$

Fig. II-21

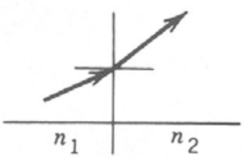
Matrices of simple optical components and systems

Free-space propagation



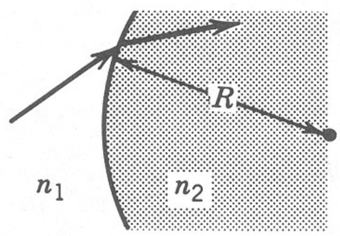
$$\mathbf{M} = \begin{bmatrix} 1 & d \\ 0 & 1 \end{bmatrix}. \tag{II-9}$$

Refraction at a planar boundary



$$\mathbf{M} = \begin{bmatrix} 1 & 0 \\ 0 & \frac{n_1}{n_2} \end{bmatrix}. \tag{II-10}$$

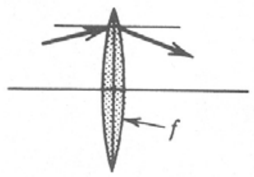
Refraction at a spherical boundary



Convex, $R > 0$; concave, $R < 0$

$$\mathbf{M} = \begin{bmatrix} 1 & 0 \\ -\frac{(n_2 - n_1)}{n_2 R} & \frac{n_1}{n_2} \end{bmatrix}. \tag{II-11}$$

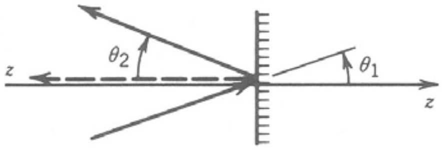
Transmission through a thin lens



Convex, $f > 0$; concave, $f < 0$

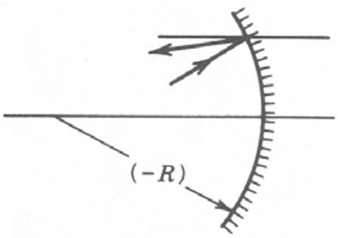
$$\mathbf{M} = \begin{bmatrix} 1 & 0 \\ -\frac{1}{f} & 1 \end{bmatrix}. \tag{II-12}$$

Reflection from a planar mirror



$$\mathbf{M} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}. \tag{II-13}$$

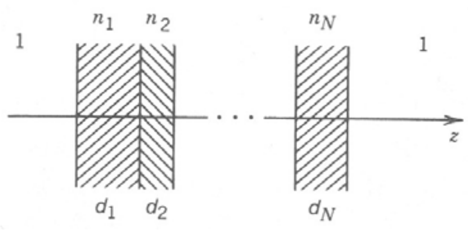
Reflection from a spherical mirror



Concave, $R < 0$; convex, $R > 0$

$$\mathbf{M} = \begin{bmatrix} 1 & 0 \\ \frac{2}{R} & 1 \end{bmatrix}. \tag{II-14}$$

Transmission through a set of transparent plates



$$\mathbf{M} = \begin{bmatrix} 1 & \sum_{i=1}^{N} \frac{d_i}{n_i} \\ 0 & 1 \end{bmatrix}. \tag{II-15}$$

Exercise: Using Eqs. II-11 and II-12 the focal length of a thin lens can be derived as

$$\frac{1}{f} = (n-1)\left(\frac{1}{R_1} - \frac{1}{R_2}\right) \tag{II-16}$$

The eikonal equation

The ray trajectories are often characterized by the surfaces to which they are normal. Let *S*(r) be a scalar function such that its equilevel surfaces, *S*(r) = const., are everywhere normal to the rays (Fig. II-22).

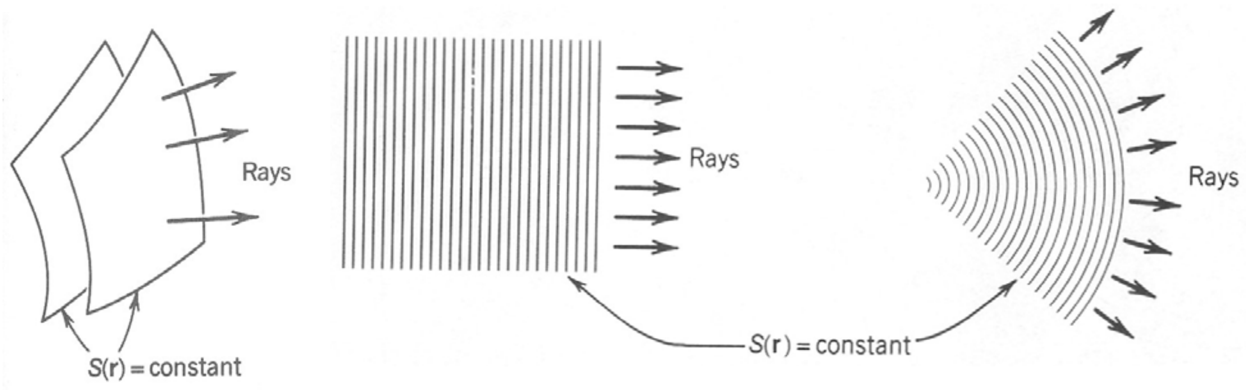


Fig. II-22

If *S*(r) is known, the ray trajectories can readily be constructed because the normal to the equilevel surfaces at a position **r** is in the direction of the gradient vector $\nabla S(\mathbf{r})$. The function *S*(r), called the eikonal, is akin to the potential function *V*(r) in electrostatics, with the role of the optical rays being played by the lines of the electric field **E** = $-\nabla V$.
To satisfy Fermat's principle, the eikonal *S*(r) must satisfy a partial differential equation known as the *eikonal equation*[1]

$$\left(\frac{\partial S}{\partial x}\right)^2 + \left(\frac{\partial S}{\partial y}\right)^2 + \left(\frac{\partial S}{\partial z}\right)^2 = n^2 \qquad \text{(II-17)}$$

$$\left|\nabla S\right|^2 = n^2 \qquad \text{(II-18)}$$

Fermat's principle can also be shown to follow from the eikonal equation. Therefore, *either the eikonal equation or Fermat's principle may be regarded as the principal postulate of ray optics*.

[1] See e.g. M. Born and E. Wolf, *Principles of Optics,* Pergamon Press, New York, 6th edition, 1980.