Lorentz invariance of physical laws and transformation of physical quantities

Some consequences of the Lorentz transformation: simultaneity is relative, Lorentz contraction, time dilatation

One of the most striking consequences of the Lorentz transformation is that simultaneity as a universal concept has to be abandoned. *Simultaneity is also relative.* This is obvious from (VII-28). Before we address this we note that the clocks placed at different positions in an inertial reference frame can be synchronized by using light for synchronisation utilizing the constancy of its speed (Fig. VII-6).

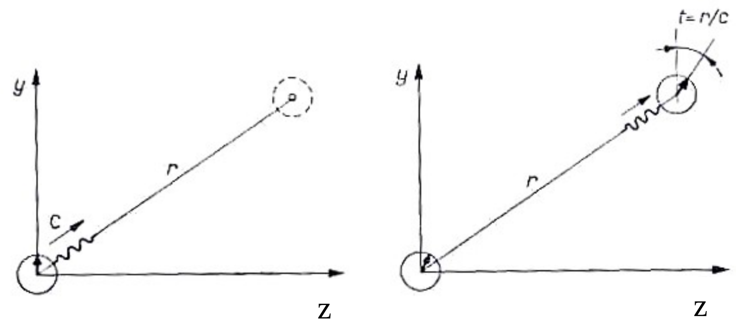

Fig. VII-6

If the clocks in *K* indicate zero, *t* = 0, it follows from (VII-28) that the clocks *K'* show different times dependent on their location:

$$t' = \gamma\left(t - \frac{v}{c^2}\,z\right) = -\gamma\,\frac{v}{c^2}\,z = -\,\frac{v}{c^2}\,z'$$

Lorentz contraction

Another consequence of the Lorentz transformation is that an observer sitting in *K* perceives the length of a moving object shorter than the length of the object in its rest frame. Let's suppose that a rod is placed in *K'* with its ends at $z_1{}' = 0$ and $z_2{}' = \ell_0$. The observer in *K* can assess the rod length ℓ in his coordinate system by determining the position of its two ends *at the same time t,*

$$\ell = z_2(t) - z_1(t)$$

Let's perform the measurement at *t* = 0. At this instant $z_1(t=0) = 0$, whereas $z_2(t=0) = \gamma(z_2{}' + vt')$. The instant *t'* can be determined from $t = \gamma(t' + \frac{v}{c^2} z_2{}') = 0 \Rightarrow t' = -\frac{v}{c^2} z_2{}'$, the substitution of which into the previous expression yields $z_2(t=0) = \gamma(z_2{}' - \frac{v^2}{c^2} z_2{}') = \frac{1}{\gamma} z_2{}' = \frac{1}{\gamma}\ell_0$, resulting in

$$\ell = z_2(t=0) - z_1(t=0) = \frac{1}{\gamma}\ell_0 \qquad \text{(VII-30)}$$

The length of the fast moving rod is perceived as contracted by the relativistic factor γ as compared to its length measured in its rest frame. This phenomenon is called *Lorentz contraction*.

The duration of an event also depends on the state of the motion of the observer. Suppose an event occurring at $z' = 0$ in *K'* starts at $t_1' = 0$ and ends at $t_2' = T_0$. An observer sitting in *K* measures its duration as

$$T = t_2 - t_1,$$

which can be determined from $t_1 = \gamma(t_1' + \frac{v}{c^2}z') = 0$ and $t_2 = \gamma(t_2' + \frac{v}{c^2}z') = \gamma t_2' = \gamma T_0$ as

$$T = t_2 - t_1 = \gamma T_0 \qquad \text{VII-31)}$$

The observer sitting in *K* perceives the duration of an event occurring in *K*' within T_0 stretched in time, by the relativistic factor γ. In other words, from the perspective of an observer sitting in *K*, the clocks in *K'* run slower, as illustrated in Fig. VII-7.

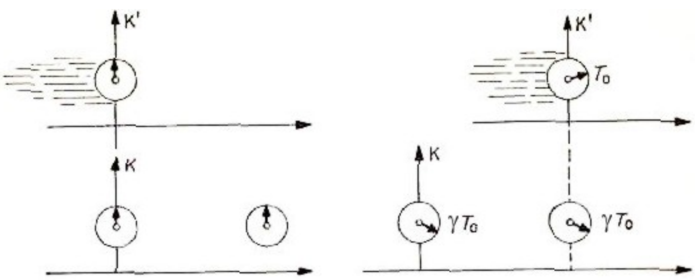

Fig. VII-7

Moving clocks run more slowly than a stationary clock, an effect that has been referred to as *time dilatation*. This paradoxical result has been verified with ultra-precise macroscopic clocks[1]. Time dilatation is also being verified routinely these days in high-energy experiments: the decay of metastable particles appears to be slowed down if they move at relativistic speeds with respect to the laboratory frame.

Relativistic Doppler shift

A light beam $E = E_0 \cos\omega\left(t - \frac{z}{c}\right)$, with a frequency ω, propagates in *K*. What is its frequency ω' in the moving frame *K'* ? To answer the question, we merely have to express the coordinates *t* and *z* with *t'* and *z'* by use of (VII-28). After some rearrangements, we obtain for the *relativistic Doppler shift*
(Exercise)

[1] J. C. Hafele and R. E. Keating, *Science* 177, 166 (1972).

$$E' = E_0' \cos \gamma\omega \left(1-\frac{v}{c}\right)\left(t'-\frac{z'}{c}\right) = E_0' \cos \omega' \left(t'-\frac{z'}{c}\right)$$

$$\omega' = \omega\gamma \left(1-\frac{v}{c}\right) = \omega\sqrt{\frac{1-\frac{v}{c}}{1+\frac{v}{c}}} \qquad \text{(VII-32)}$$

Addition of velocities

Since the measurement of length as well as time depends on the state of motion of the observer, velocities do not simply add as it was the case in Galilean relativity. Suppose that there is a moving point in *K'* whose velocity is $u_z{}'$. What velocity u_z will an observer measure in *K*? According to the definition of velocity

$$u_z = \frac{dz}{dt} = \frac{dz}{dt'}\,\frac{dt'}{dt}$$

$$\frac{dz}{dt'} = \gamma\frac{d}{dt'}(z'+vt') = \gamma(u_z'+v) \qquad \text{(VII-33)}$$

$$\frac{dt'}{dt} = \gamma\frac{d}{dt}(t-\frac{v}{c^2}z) = \gamma\left(1-\frac{v}{c^2}u_z\right)$$

yielding

$$u_z = \gamma^2\,(u_z'+v)\left(1-\frac{v}{c^2}u_z\right) \qquad \text{(VII-34)}$$

from which we obtain

$$u_z = \frac{u_z'+v}{1+\frac{u_z'v}{c^2}} \qquad \text{(VII-35a)}$$

If u' is of arbitrary direction, the same procedure leads to

$$u_x = \frac{u_x'}{\gamma\left(1+\frac{u_z'v}{c^2}\right)} \quad ; \quad u_y = \frac{u_y'}{\gamma\left(1+\frac{u_z'v}{c^2}\right)} \qquad \text{(VII-35b,c)}$$

The first postulate dictates that inversion of (VII-35) results in expressions of the same form except for a change in the sign of *v*. As a matter of fact, inverting e.g. (VII-35a) leads to

$$u'_z = \frac{u_z - v}{1 - \frac{u_z v}{c^2}} \qquad \text{(VII-35a')}$$

in accordance with the first postulate. For *v/c* << 1, (VII-35) goes over to u = u' + v as dictated by the Galilean coordinate transformation. For $u_z{}' = c$ (VII-35) yields $u_z = c$. The speed of light can not be overcome.

Relativistic invariant expressions for momentum and energy of a particle

We have seen at the beginning of this chapter that the form of Newton's equation of motion (VII-2,3) constituting the law of mechanics, is preserved in Galilean relativity. However, it is not invariant under the Lorentz transformation. If the Lorentz transformation constitutes the proper connection between the space-time coordinates of inertial reference frames, *the laws of mechanics need to be corrected*, so that they become invariant under the Lorentz transformation.

Consider the perfectly elastic collision of two balls of masses $m_1{}' = m_2{}' = m'$ moving with $u_1{}' = u'$ and $u_2{}' = -u'$ in reference frame *K'* (Fig. VII-8). At the instant of the collision their velocities in *K'* become zero, before they bounce back after the collision with $u_1{}'^{*} = -u'$ and $u_2{}'^{*} = u'$.

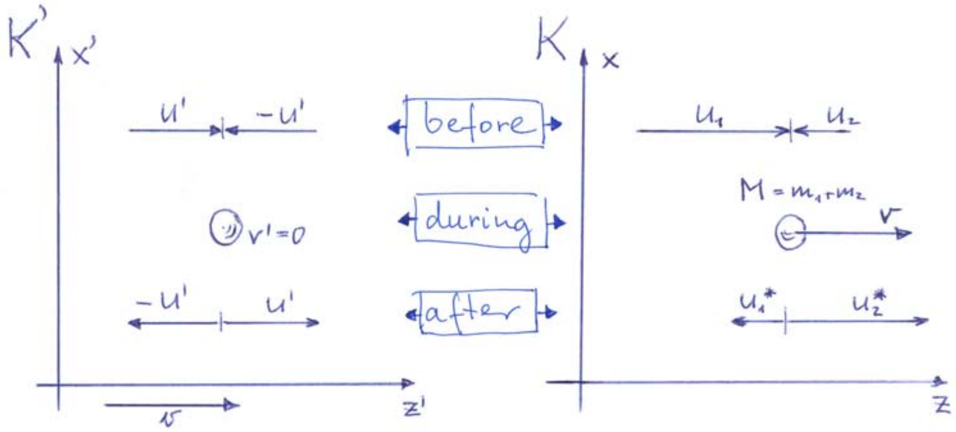

Fig. VII-8

In correcting/generalizing the laws of mechanics, *we require the conservation of momentum and mass* in any inertial frame of reference, in the spirit of the first postulate. These conservation laws must apply during all stages (before, during, after, see Fig. VII-8) of the collision. This requirement implies for the quantities measured by an observer resting in *K*

$$m_1 u_1 + m_2 u_2 = M v \qquad ; \qquad M = m_1 + m_2$$

$$u_1 = \frac{u' + v}{1 + \frac{u' v}{c^2}} \qquad ; \qquad u_2 = \frac{-u' + v}{1 - \frac{u' v}{c^2}} \tag{VII-36}$$

From these equations we obtain (Exercise)

$$\frac{m_1}{m_2} = \frac{1 + \frac{u' v}{c^2}}{1 - \frac{u' v}{c^2}} = \frac{v - u_2}{u_1 - v} = \frac{1 - \frac{u_2 v}{c^2}}{1 - \frac{u_1 v}{c^2}} = \frac{\sqrt{1 - u_2^2 / c^2}}{\sqrt{1 - u_1^2 / c^2}} \tag{VII-37}$$

implying that the mass of a moving particle/body is increased as compared to its rest mass m_0 according to

$$m = \frac{m_0}{\sqrt{1 - \frac{u^2}{c^2}}} \tag{VII-38}$$

This can be reformulated by defining *K'* as the frame co-moving with the particle, in which the particle is at rest and is of mass m_0 (we call this the rest frame of the particle) to determine the mass in *K*, in which *K'* (and the particle) moves with *v*

$$m = \frac{m_0}{\sqrt{1 - \frac{v^2}{c^2}}} = \gamma m_0 \tag{VII-38'}$$

From this and the first equation in (VII-37) follows that the Lorentz transformation law of mass reads as

$$m = \gamma \left(1 + \frac{u' v}{c^2} \right) m'; \qquad m' = \gamma \left(1 - \frac{u v}{c^2} \right) m \tag{VII-39}$$

where *m'* and *m* are the masses measured in *K'* and *K*, respectively, *v* is the speed of *K'* with respect to *K*, *u'* and *u* are the speed of the body in *K'* and *K*, respectively, and all speeds are assumed to be directed along the *z* axis.

From the Lorentz transformation law of velocities and the requirement of momentum and mass conservation it follows that the mass of a body is dependent on its velocity (VII-38) and *is not invariant under the Lorentz transformation* (VII-39).

The equation of motion of relativistic mechanics can now be written as

$$\frac{d\mathbf{p}}{dt} = \mathbf{F}$$
$$\mathbf{p} = m\mathbf{u} = \frac{m_0}{\sqrt{1 - \frac{u^2}{c^2}}}\,\mathbf{u} \qquad \text{(VII-40)}$$

Equations (VII-40) implies that the work done on a particle by force *F* can be expressed as

$$dW_{\mathrm{kin}} = \mathbf{F}\, d\mathbf{r} = \frac{d(m\mathbf{u})}{dt}\, d\mathbf{r} = \mathbf{u}\, d(m\mathbf{u}) \qquad \text{(VII-41)}$$

Assuming that d**u** is parallel to **u**, we may write

$$dW_{\mathrm{kin}} = \mathbf{u}(\mathbf{u}dm + m\,d\mathbf{u}) = u^2 dm + m\,\mathbf{u}d\mathbf{u} = c^2 dm\left(\frac{u^2}{c^2} + \frac{mu}{c^2}\frac{du}{dm}\right) \qquad \text{(VII-42)}$$

From (VII-38)

$$\frac{du}{dm} = \left(\frac{dm}{du}\right)^{-1} = \frac{c^2 - u^2}{mu} \qquad \text{(VII-43)}$$

which, upon substitution in (VII-42), yields

$$dW_{\mathrm{kin}} = c^2 dm; \qquad W_{\mathrm{kin}} = mc^2 - m_0 c^2 \qquad \text{(VII-44)}$$

In the limit of *v/c* << 1, (VII-44) and (VII-38) yields $W_{kin} \approx (1/2) m_0 u^2$ (Exercise).

From the Lorentz transformation law of velocities and the requirement of momentum and mass conservation we have now also derived that the change in kinetic energy of a particle is proportional to that of its mass.

Einstein interpreted (VII-44) more generally. He referred to m_0c^2 as the *rest energy* of a body of rest mass m_0, implying that the total energy of the particle/body is simply given by

$$W = mc^2 = \gamma\, m_0\, c^2 = \sqrt{c^2 p^2 + m_0^2 c^4} \qquad \text{(VII-45)}$$

where m_0 is the mass of the particle/body in its rest frame of reference *K'* moving with the speed *v* of the particle with respect to the laboratory frame *K* and γ is the relativistic factor. Using the transformation rules for the mass (VII-39) and velocity (VII-35) it can be readily shown that

$$(m_0 c)^2 = \frac{W^2}{c^2} - p^2 \qquad \text{(VII-46)}$$

which is recognized as the square of the "length" of the energy-momentum 4-vector ($p_0 = W/c$, p), p_0^2 - p^2, is invariant under the Lorentz transformation, by showing that $(m_0 c)^2 = m^2(c^2 - u^2) = m'^2(c^2 - u'^2)$ (Exercise).

Equation (VII-45) expresses the *equivalence of mass and energy of matter*. At the beginning of this section we required the conservation of momentum and mass for deriving the relativistically correct laws of mechanics. Now it is clear that this is equivalent to requiring the conservation of momentum and energy. The equivalence of mass and energy has been confirmed in numerous beautiful nuclear physics and particle physics experiments meanwhile. In nuclear fission the energy released is found to be equal to the difference of the mass of the initial fuel and that of the final fission products.

Transformation of the force

We require the definition of the force to be valid in any inertial frame

$$\mathrm{F} = \frac{d\mathrm{p}}{dt} = \frac{d(m\mathrm{u})}{dt} \quad ; \quad \mathrm{F}' = \frac{d\mathrm{p}'}{dt'} = \frac{d(m'\mathrm{u}')}{dt'} \qquad \text{(VII-47,47')}$$

Since we know how to transform the momentum and time, from (VII-47 and 47') we can determine the Lorentz transformation law of the force. To this end we write

$$F_z = \frac{d}{dt} p_z = \frac{dp_z}{dt'} \frac{dt'}{dt} = \left(\frac{dt}{dt'}\right)^{-1} \frac{d}{dt}(mu_z) \qquad \text{(VII-48)}$$

where from (VII-28)

$$\frac{dt}{dt'} = \gamma\left(1 + \frac{v}{c^2} u'_z\right) \qquad \text{(VII-49)}$$

By using the transformation laws for u_z (VII-35a) and m (VII-39) we obtain (Exercise)

$$F_z = \frac{1}{1 + \frac{u'_z v}{c^2}} \frac{d}{dt'}(m'u'_z + m'v) = \frac{1}{1 + \frac{u'_z v}{c^2}} \left(\frac{dp'_z}{dt'} + \frac{v}{c^2}\frac{dW'}{dt'}\right) \qquad \text{(VII-50)}$$

Now by using (VII-47') and $F_x' u_x' + F_y' u_y' + F_z' u_z' = dW'/dt'$ we arrive at our final result

$$F_z = F_z' + \frac{v}{c^2 + u_z' v}(u_x' F_x' + u_y' F_y') \qquad \text{(VII-51a)}$$

The same procedure yields

$$F_x = \frac{F_x'}{\gamma\left(1+\frac{u_z' v}{c^2}\right)} \qquad ; \qquad F_y = \frac{F_y'}{\gamma\left(1+\frac{u_z' v}{c^2}\right)} \qquad \text{(VII-51b,c)}$$

As always, the inverse transformation is obtained by interchanging the primed and unprimed variables and reversing the sign of *v*. (VII-51) reveals, that the transverse force components are weaker in the laboratory frame as compared to the rest frame of the particle, whereas the longitudinal component is left unchanged (for $u_x', u_y' = 0$).

Knowing the transformation rules of the force (from the invariance of the generalized Newton equation VII-47) and those of the velocities (VII-35, derived directly from the Lorentz transformation) and the electromagnetic fields (from the invariance of Maxwell's equations), we find that *the expression of the electromagnetic (Lorentz) force acting on a charge q is invariant under the Lorentz transformation with q' = q*

$$\mathrm{F} = q(\mathrm{E} + \mathrm{u} \times \mathrm{B}); \qquad \mathrm{F}' = q(\mathrm{E}' + \mathrm{u}' \times \mathrm{B}') \qquad \text{(VII-52,52')}$$

Transformation of electromagnetic fields

The transformation laws for the transverse field components E_x and B_y have been determined before (VII-25), the other transverse components E_y and B_x can be obtained from the same procedure upon changing the polarization of the plane wave ansatz (VII-15). In this way, we arrive at the transformation rules for the transverse field components

$$E_x' = \gamma(E_x - vB_y) \qquad ; \qquad B_x' = \gamma(B_x + \frac{v}{c^2}E_y) \qquad \text{(VII-53x)}$$

$$E_y' = \gamma(E_y + vB_x) \qquad ; \qquad B_y' = \gamma(B_y - \frac{v}{c^2}E_x) \qquad \text{(VII-53y)}$$

The transformation rule for the longitudinal (*z*) components can also be determined from the requirement of the invariance of Maxwell's equations. However, it is more straightforward to derive these transformations from the invariance of the Lorentz force. To this end consider a charge *q* resting at the origin of frame *K'* moving with *v* along the *z* direction of frame *K* (Fig. VII-9).

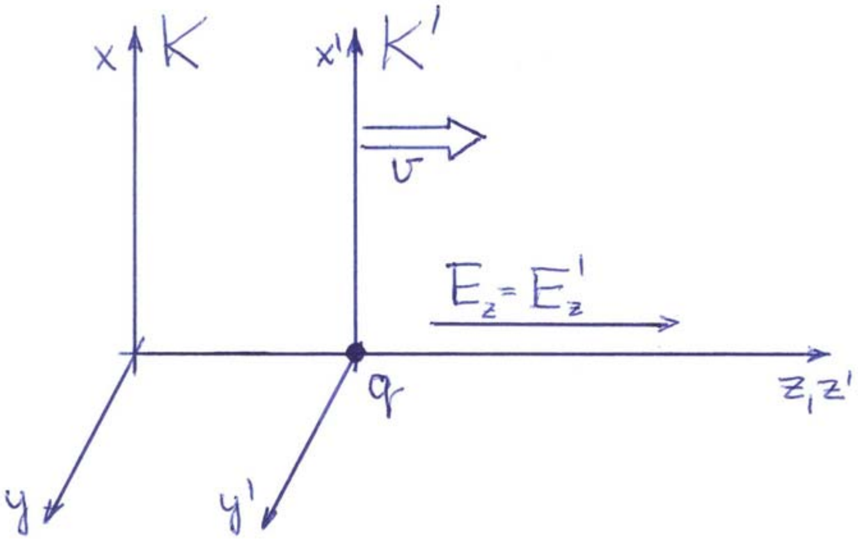

Fig. VII-9

As it is at rest, only the electric field exerts a force *F' = qE'* and according to (VII-51a) only the *z'* component of this force contributes to the *z* component of the transformed force acting in *K*

$$F_z = F'_z \quad ; \quad F_z = qE_z \quad ; \quad F'_z = qE'_z$$

from which

$$E'_z = E_z$$

In order to probe the *z'* component of the magnetic field, let us now have a probing charge *q* moving with $u_y{}'$ in *K'* along the *y'* axis of this frame (Fig. VII-10):

$$u'_x = 0 \quad ; \quad u'_y \neq 0 \quad ; \quad u'_z = 0$$

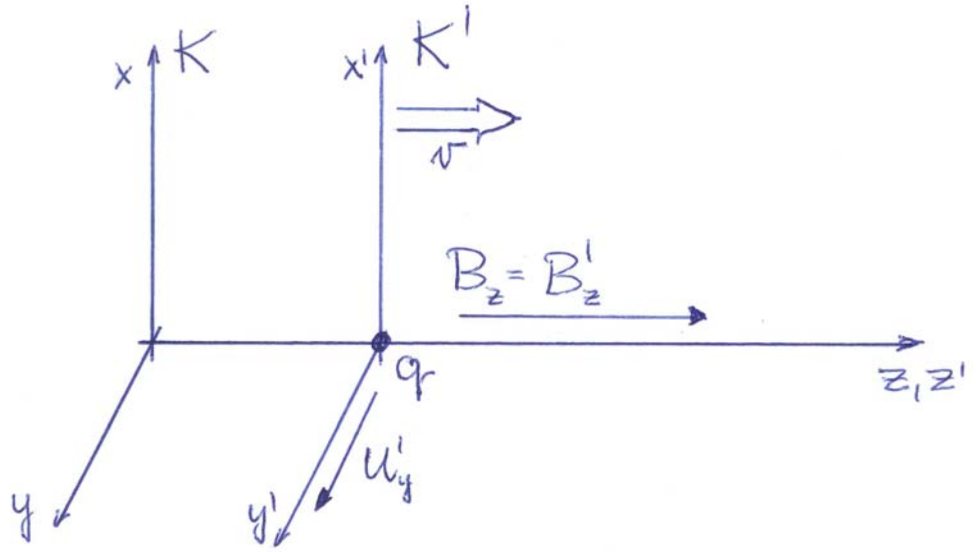

Fig. VII-10

Suppose in *K'* we have a magnetic field **B'**. We consider only the x', x -component of the force that is proportional to u_y' and u_y, respectively, in order to probe the *z'* and *z* component of the magnetic field, respectively. For these force components we have, according to (VII-51b)

$$F_x = \frac{F_x'}{\gamma} \quad ; \quad F_x = q\, u_y\, B_z \quad ; \quad F_x' = q\, u_y'\, B_z'$$

which, together with the velocity transformation $u_y = u_y' / \gamma$ (from Eq. VII-35c), yields

$$B_z' = B_z$$

As a result, for the longitudinal field components we have

$$E_z' = E_z \quad ; \quad B_z' = B_z \tag{VII-53z}$$

We can now summarize the Lorentz transformation rules for the electromagnetic fields as follows

$$\begin{aligned} &\mathbf{E}_{||}' = \mathbf{E}_{||} \quad ; \quad \mathbf{B}_{||}' = \mathbf{B}_{||} \\ &\mathbf{E}_{\perp}' = \gamma(\mathbf{E}_{\perp} + \mathbf{v}\times\mathbf{B}_{\perp}) \quad ; \quad \mathbf{B}_{\perp}' = \gamma\left(\mathbf{B}_{\perp} - \frac{1}{c^2}\,\mathbf{v}\times\mathbf{E}_{\perp}\right) \end{aligned} \tag{VII-53}$$

Relativistic invariance of charge

So far we have disregarded sources (charge and current) in Maxwell's equations. Introducing these source terms in the equations and requiring that the equations preserve their form upon the Lorentz transformation, we obtain the transformation rules for the source terms

$$\begin{aligned} &J_x' = J_x \;;\; J_y' = J_y \;;\; J_z' = \gamma\,(J_z - v\rho) \qquad \Big| \qquad J_z = \gamma\,(J_z' + v\rho') \\ &\rho' = \gamma\left(\rho - \frac{v}{c^2}\,J_z\right) \qquad \Big| \qquad \rho = \gamma\left(\rho' + \frac{v}{c^2}\,J_z'\right) \end{aligned} \tag{VII-54,54'}$$

We can now verify the invariance of charge under the Lorentz transformation. Suppose an infinitesimal volume d*V'* = d*x'*d*y'*d*z'* in *K'* is filled with a charge of density ρ', which is at rest in the frame *K'*, i.e. **J'** =0. We now establish the connection between the charge contained in this volume as measured by an observer in *K'*, *q'*= ρ'd*V'*, and by an observer resting in *K*, *q* = ρd*V*. In doing so, we draw on the transformation of ρ' and d*V'*

$$q = \rho\, dx\, dy\, dz \qquad ; \qquad q' = \rho' dx' dy' dz'$$

$$dx = dx' \quad ; \quad dy = dy' \quad ; \quad dz = \frac{1}{\gamma} dz' \qquad \text{(VII-55)}$$

$$\rho = \gamma \left(\rho' + \frac{v}{c^2} J'_z \right) \quad ; \quad J'_z = 0$$

From which we confirm the invariance of charge

$$q = q' \qquad \text{(VII-56)}$$

Fields of a moving charge

Consider a charge resting at the origin of *K'* moving with *v* along the *z* axis of *K* (Fig. VII-11). We wish to determine the field of this charge as measured by an observer resting at the point *P* in *K.*

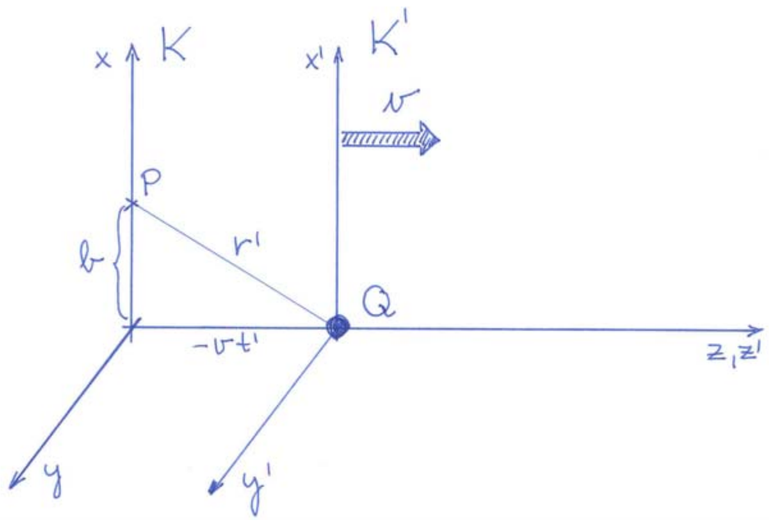

Fig. VII-11

As the charge rests in *K',* we have only static electric fields in *K'*

$$E'_x = \frac{Q}{4\pi\varepsilon_0}\frac{b}{r'^3} \quad ; \quad E'_y = 0 \quad ; \quad E'_z = \frac{Q}{4\pi\varepsilon_0}\frac{-vt'}{r'^3}$$

$$B'_x = 0 \quad ; \quad B'_y = 0; \quad B'_z = 0$$

$$\left.\begin{array}{l} r' = \left(b^2 + v^2 t'^2\right)^{1/2} \\ \left.\begin{array}{l} t' = \gamma\left(t - \dfrac{v}{c^2}z\right) \\ z = 0 \end{array}\right\} \Rightarrow t' = \gamma t \end{array}\right\} \Rightarrow r' = (b^2 + \gamma^2 v^2 t^2)^{1/2} \qquad \text{(VII-57)}$$

From these we readily obtain the transformed fields by using (VII-53) as

$$E_x = \gamma(E'_x + vB'_y) = \gamma E'_x$$

$$E_y = \gamma(E'_y + vB'_x) = 0$$

$$E_z = E'_z$$

$$B_x = \gamma\left(B'_x - \frac{v}{c^2}E'_y\right) = 0$$

$$B_y = \gamma\left(B'_y + \frac{v}{c^2}E'_x\right) = \gamma\,\frac{v}{c^2}\,E'_x \qquad \text{(VII-58)}$$

$$B_z = B'_z = 0$$

resulting in

$$E_x = \frac{Q}{4\pi\varepsilon_0} \frac{\gamma b}{(b^2 + \gamma^2 v^2 t^2)^{3/2}}$$

$$E_z = -\frac{Q}{4\pi\varepsilon_0} \frac{\gamma v t}{(b^2 + \gamma^2 v^2 t^2)^{3/2}} \qquad \text{(VII-59)}$$

$$B_y = \frac{Q}{4\pi\varepsilon_0} \frac{v}{c^2} \frac{\gamma b}{(b^2 + \gamma^2 v^2 t^2)^{3/2}} = \frac{v}{c^2} E_x$$

In K there is observed a magnetic field, which – for nonrelativistic velocities – can be approximately written as

$$\mathbf{B} \approx \frac{1}{\varepsilon_0 c^2} \frac{Q}{4\pi} \frac{\mathbf{v} \times \mathbf{r}}{r^3} \qquad \text{(VII-60)}$$

Hence the Lorentz transformation of the electrostatic field of a moving charge yields the *Biot-Savart expression* for the magnetic field of a moving charge.

The electric and magnetic fields of a relativistic charged particle are depicted in Fig. VII-12.

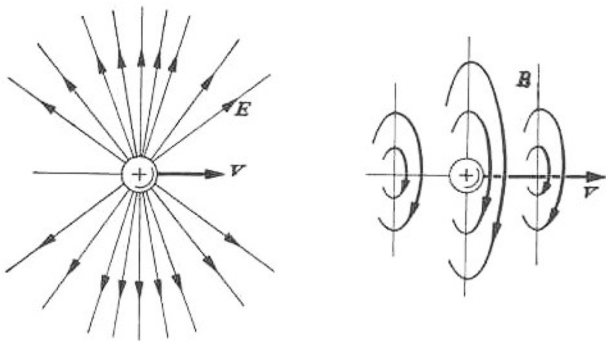

Fig. VII-12

In the limit of $b \ll r$ the longitudinal component of the electric field

$$E_z(b \ll r) \approx -\frac{Q}{4\pi\varepsilon_0} \frac{\gamma v t}{(\gamma v t)^3} = -\frac{Q}{4\pi\varepsilon_0} \frac{1}{r^2} \frac{1}{\gamma^2} \qquad \text{(VII-61)}$$

is reduced by a factor of γ^2 as compared to the field strength of a charge at rest. It is also interesting to note that a charge co-moving with Q at a transverse distance b experiences an electric and magnetic force (Fig. VII-12)

$$F_{x,E} = q\,E_x$$

$$F_{x,B} = q\,v\,B_y = -q\,\frac{v^2}{c^2}\,E_x \qquad \text{(VII-62)}$$

which, for $v \rightarrow c$ become nearly equal in magnitude and tend to compensate each other (Fig. VII-13). As a consequence of (VII-61) and (VII-62) the Coulomb repulsion in a relativistic bunch of particles of identical charge (electrons, protons) is dramatically reduced.

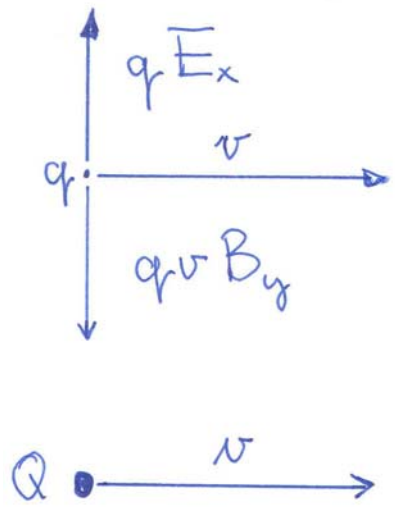

Fig. VII-13